

COXETER GROUPS ARE NOT HIGHER RANK ARITHMETIC GROUPS

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ABSTRACT. Let W be an irreducible finitely generated Coxeter group. The geometric representation of W in $GL(V)$ provides a discrete embedding in the orthogonal group of the Tits form (the associated bilinear form of the Coxeter group). If the Tits form of the Coxeter group is non-positive and non-degenerate, the Coxeter group does not contain any finite index subgroup isomorphic to an irreducible lattice in a semisimple group of \mathbb{R} -rank ≥ 2 .

1. INTRODUCTION

Let $S = \{s_1, s_2, \dots, s_n\}$ be a finite set and W be a group generated by S with the relations

$$(s_i s_j)^{m_{i,j}} = 1,$$

where $m_{i,i} = 1$, $\forall 1 \leq i \leq n$ and $m_{i,j} \in \{2, 3, \dots, \infty\}$, $\forall i \neq j$. The group W is called the Coxeter group. The Coxeter system (W, S) is called irreducible if the Coxeter graph ([4], Section 2.1) is connected. Now we define a symmetric bilinear form (Tits form) B on a vector space V of dim n over \mathbb{R} , with a basis $\{e_1, e_2, \dots, e_n\}$ in one-to-one correspondence with S as

$$B(e_i, e_j) = -\cos\left(\frac{\pi}{m_{i,j}}\right), \quad \forall 1 \leq i, j \leq n.$$

(This expression is interpreted to be -1 in case $m_{i,j} = \infty$.)

For each $s_i \in S$ we can now define a reflection $\sigma_i : V \rightarrow V$ by the rule:

$$\sigma_i \lambda = \lambda - 2B(e_i, \lambda)e_i.$$

Clearly $\sigma_i e_i = -e_i$, while σ_i fixes $H_i = \{v \in V | B(v, e_i) = 0\}$ pointwise. In particular, we see that σ_i has order 2 in $GL(V)$. The bilinear form B is preserved by all of the elements σ_i hence it will be preserved by each element of the subgroup of $GL(V)$ generated by the $\sigma_i (1 \leq i \leq n)$.

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Proposition 1.1 ([4], Proposition 5.3). *There is a unique homomorphism $\sigma : W \rightarrow \mathrm{GL}(V)$ sending s_i to σ_i , and the group $\sigma(W)$ preserves the form B on V . Moreover, for each pair $s_i, s_j \in S$, the order of $s_i s_j$ in W is precisely $m_{i,j}$.*

Proposition 1.2 ([4], Corollary 5.4). *The representation $\sigma : W \rightarrow \mathrm{GL}(V)$ is faithful.*

Relative to the basis $\{e_1, e_2, \dots, e_n\}$ of V , we can identify V with \mathbb{R}^n and $\mathrm{GL}(V)$ with $\mathrm{GL}(n, \mathbb{R})$, the latter in turn being viewed as an open set in \mathbb{R}^{n^2} .

Proposition 1.3 ([4], Proposition 6.2). *$\sigma(W)$ is a discrete subgroup of $\mathrm{GL}(V)$, topologized as above.*

In this paper we will assume that the Coxeter system (W, S) is irreducible and the Tits form B is non-degenerate and the Coxeter group W is infinite. By the above proposition, W is a discrete subgroup of the corresponding orthogonal group $G := \mathrm{O}(B)(\mathbb{R})$. G is a real Lie group, with a Haar measure which provides a notion of volume ν for $W \backslash G$, the homogeneous space of right cosets of G with respect to W . Let $C := \{v \in V \mid B(v, e_i) > 0, \forall 1 \leq i \leq n\}$. The goal of this paper is to prove Theorem 1.4 (stated below) which has been proved in [3] also, by using a different technique. In [3], it has been proved that an infinite Coxeter group has a subgroup of finite index which admits a homomorphism onto \mathbb{Z} .

Theorem 1.4. *If W is an irreducible finitely generated Coxeter group with the non-positive and non-degenerate Tits form, then it does not contain any finite index subgroup isomorphic to an irreducible lattice in a connected semisimple Lie group without non-trivial compact factor groups, of real rank ≥ 2 .*

In fact more is true:

Theorem 1.5. (a) *If W is an irreducible finitely generated Coxeter group with the non-positive and non-degenerate Tits form, then it does not contain any finite index subgroup isomorphic to a higher rank S -arithmetic group (i.e., lattice in a product of Lie groups and p -adic groups).*

For example, the Coxeter group W does not contain any finite index subgroup isomorphic to $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$.

(b) *More generally, if k_1, k_2, \dots, k_r are local fields and G_1, G_2, \dots, G_r are semisimple algebraic groups defined over k_1, k_2, \dots, k_r respectively such that each G_i has k_i -rank ≥ 1 and $\sum_{i=1}^r k_i\text{-rank}(G_i) \geq 2$, then W*

does not contain any finite index subgroup isomorphic to an irreducible lattice Γ in $\prod_{i=1}^r G_i(k_i)$.

For example, the Coxeter group W does not contain any finite index subgroup isomorphic to $SL_3(\mathbb{F}_p[t])$ in $SL_3(\mathbb{F}_p((\frac{1}{t})))$.

Theorem 1.5 can be proved by the same method used for the proof of Theorem 1.4 using Theorem 1.6 (stated below) and the superrigidity of lattices in semisimple groups over local fields of arbitrary characteristic (see [6]; cf. [9]). Therefore, in this paper we will prove Theorem 1.4; and for the sake of completeness of the proof we will also prove the following theorem (stated in [2] as an exercise) and its consequences:

Theorem 1.6 (§12, Exercise § 4 of Chapter V in [2]). *If W is a lattice in $O(B)(\mathbb{R})$, then B has signature $(n-1, 1)$ and $B(v, v) < 0$, for all $v \in \mathbb{C}$.*

It is known that a Coxeter group W can not be a lattice in $O(B)(\mathbb{R}) = O(n-1, 1)$, for $n > 10$ (17, Exercise § 4 of Chapter V in [2]).

To prove Theorem 1.4 we will use the following theorem of G. A. Margulis:

Theorem 1.7 (Theorem 6.16 of Chapter IX in [6]). *Let H be a connected semisimple Lie group without nontrivial compact factor groups. Let $\Gamma \subset H$ be a lattice, k a local field, F a connected semisimple k -group, and $\delta : \Gamma \rightarrow F(k)$ a homomorphism such that the subgroup $\delta(\Gamma)$ is Zariski dense in F . Assume that $\text{rank } H \geq 2$ and the lattice Γ is irreducible. Then,*

- (a) *for k isomorphic neither to \mathbb{R} nor to \mathbb{C} , i.e. for non-archimedean k , the subgroup $\delta(\Gamma)$ is relatively compact in $F(k)$.*
- (b) *for $k = \mathbb{R}$, if the group F is adjoint and has no nontrivial \mathbb{R} -anisotropic factors, then δ extends, uniquely, to a continuous homomorphism $\tilde{\delta} : H \rightarrow F(\mathbb{R})$.*

In this paper (Section 3) we will also show that a right angled Coxeter group W generated by 3 elements is isomorphic to a lattice in the group $O(B)(\mathbb{R}) = O(2, 1)$ of real rank 1.

2. PROOF OF THEOREM 1.6

The proof has been sketched in the Bourbaki exercise (§12, Exercise § 4 of Chapter V in [2]), and for the sake of completeness we fill in the details.

Let V^* be the dual of V . Since W acts on V via σ , it also acts, by transport of structure, on V^* . The corresponding representation

$$\sigma^* : W \rightarrow GL(V^*)$$

is called the contragredient representation of σ . We have

$$\sigma^*(w) = \sigma^T(w^{-1}), \text{ for all } w \in W.$$

If $x^* \in V^*$ and $w \in W$, we denote by $w(x^*)$ the transform of x^* by $\sigma^*(w)$.

If $s_i \in S$, denote by A_i the set of $x^* \in V^*$ such that $x^*(e_i) > 0$. Let C be the intersection of the A_i , $1 \leq i \leq n$. When S is finite, C is an open set in V^* . The following theorem and its corollaries are from [2]:

Theorem 2.1 (Tits). *If $w \in W$ and $C \cap wC \neq \emptyset$, then $w = 1$.*

Corollary 2.2. *The representations σ and σ^* are injective.*

Corollary 2.3. *If S is finite, $\sigma(W)$ is a discrete subgroup of $GL(V)$ (provided with its canonical Lie group structure); similarly, $\sigma^*(W)$ is a discrete subgroup of $GL(V^*)$.*

Let G be a closed subgroup of $GL(V)$ containing W . Let G be unimodular and D be a half line of V^* contained in C i.e., $D = \mathbb{R}_{>0}v \subset C$, for some $v \in C$, and let G_D be the stabilizer of D in G .

Lemma 2.4. *Let Δ be the set of elements $g \in G$ such that $g(D) \subset C$. Then Δ is open, stable under right multiplication by G_D , and that the composite map $\Delta \rightarrow G \rightarrow W \backslash G$ is injective, $W \backslash G$ denoting the homogeneous space of right cosets of G with respect to W .*

Proof. First, we show that Δ is open in G . For, $\Delta = \{g \in G \mid g(v) \in C\}$, where $v \in V^*$ such that $D = \mathbb{R}_{>0}v \subset C$. We define a map $f : G \rightarrow V^*$ by $g \mapsto g(v)$. It is clear that f is continuous and C is open in V^* , hence $f^{-1}(C) = \Delta$ is open in G . Now we show that Δ is stable under right multiplication by G_D . For, let $h \in G_D$ and $g \in \Delta$.

$$gh(v) = g(\alpha v) = \alpha g(v) \in C, \text{ for some } \alpha \in \mathbb{R}_{>0}.$$

and this shows that $gh \in \Delta$. Finally, we show that the composite map $\Delta \rightarrow G \rightarrow W \backslash G$ is injective. For, let $g_1, g_2 \in \Delta$ such that $Wg_1 = Wg_2$ i.e., $g_1g_2^{-1} \in W$. Since $g_2(D) \subset C$, $D \subset g_2^{-1}(C)$. That is, $g_1(D) \subset g_1g_2^{-1}(C)$. Also, $g_1(D) \subset C$, therefore $g_1g_2^{-1}(C) \cap C \neq \emptyset$. Hence by Theorem 2.1, we get $g_1g_2^{-1} = 1$. This shows that the composite map $\Delta \rightarrow G \rightarrow W \backslash G$ is injective. \square

Lemma 2.5. *Let μ be a Haar measure on G . If $\mu(\Delta)$ is finite, the subgroup G_D is compact.*

Proof. Since Δ is an open set containing the identity element of G and the group G is locally compact, $\exists K$ a compact neighbourhood of the identity element contained in Δ . We claim that there exist finitely

many elements $h_i \in G_D$ such that every set of the form Kh , with $h \in G_D$, meets one of the Kh_i . For, suppose on the contrary that for any finite collection $\mathcal{H}_j = \{h_i\}_{i=1}^j$ of elements in G_D , $\exists h_{j+1} \in G_D$ such that $Kh_{j+1} \cap (\cup_{i=1}^j Kh_i) = \emptyset$. We start with $\mathcal{H}_1 = \{h_1\}$ in G_D . There exists $h_2 \notin \mathcal{H}_1$ in G_D such that $Kh_2 \cap Kh_1 = \emptyset$. Now take $\mathcal{H}_j = \{h_1, h_2, \dots, h_j\}$; $\exists h_{j+1} \notin \mathcal{H}_j$ in G_D such that $Kh_{j+1} \cap (\cup_{i=1}^j Kh_i) = \emptyset$. By induction on k , we get $\forall k \in \mathbb{N}$, $\exists h_{j+k} \notin \mathcal{H}_{j+k-1}$ in G_D , such that $Kh_{j+k} \cap (\cup_{i=1}^{j+k-1} Kh_i) = \emptyset$. In particular, $Kh_{j+k} \cap Kh_{j+k'} = \emptyset$, $\forall k \neq k'$. Since Δ is stable under right multiplication by any element of G_D , we get $Kh \subset \Delta$, $\forall h \in G_D$. Hence

$$\mu(\Delta) \geq \mu(\cup_{i=1}^{\infty} Kh_{j+i}) = \sum_{i=1}^{\infty} \mu(Kh_{j+i}) = \sum_{i=1}^{\infty} \mu(K) = \infty$$

(Since G is unimodular and K contains an open subset of G and hence $\mu(K) > 0$), which is a contradiction to the given hypothesis. Therefore $\exists \mathcal{H}_r = \{h_1, h_2, \dots, h_r\}$ a finite collection of elements in G_D such that $\forall h \in G_D$, $Kh \cap Kh_i \neq \emptyset$, for some $i \in \{1, 2, \dots, r\}$, which shows that $G_D \subset \cup_{i=1}^r K^{-1}Kh_i$ and hence G_D is compact (since G_D is a closed subset of G and $G_D \subset \cup_{i=1}^r K^{-1}Kh_i$ which is compact). \square

Lemma 2.6. *Let ν be a non-zero positive measure on $W \backslash G$ invariant under G . If $\nu(W \backslash G) < \infty$, then G_D is compact.*

Proof. G is unimodular with a Haar measure μ and ν is a non-zero positive measure on $W \backslash G$ invariant under G . Let ν' be a Haar measure on W . Since W is a discrete subgroup of $GL(V)$, ν' is actually the counting measure (up to a scalar multiple) on W . We prove here that $\mu(\Delta) < \infty$, which proves that G_D is compact, using the last lemma.

We have a relation in μ , ν and ν' as

$$\int_G f d\mu = \int_{W \backslash G} \left(\int_W f(wg) d\nu'(w) \right) d\nu(Wg), \quad \forall f \in \mathcal{C}_c(G). \quad (1)$$

We show that $\mu(\Delta) < \infty$, by using the above relation (1). Let the symbol $f \prec \Delta$ means that $f \in \mathcal{C}_c(G)$ with $0 \leq f \leq 1$ and support of f is contained in Δ . Since Δ is open in G , we get

$$\mu(\Delta) = \sup \left\{ \int_G f d\mu : f \prec \Delta \right\}. \quad (2)$$

Let $f \prec \Delta$. By (1), we get

$$\begin{aligned} \int_G f \, d\mu &= \int_{W \setminus G} \left(\int_W f(wg) \, d\nu'(w) \right) d\nu(Wg) \\ &\leq \int_{W \setminus G} \left(\int_W \chi_\Delta(wg) \, d\nu'(w) \right) d\nu(Wg). \end{aligned} \quad (3)$$

Since $wg \in \Delta \Leftrightarrow w \in \Delta g^{-1}$, we get

$$\begin{aligned} \int_{W \setminus G} \left(\int_W \chi_\Delta(wg) \, d\nu'(w) \right) d\nu(Wg) &= \int_{W \setminus G} \nu'(\Delta g^{-1} \cap W) d\nu(Wg) \\ &= \int_{W \setminus G} \#\{\Delta g^{-1} \cap W\} d\nu(Wg) \end{aligned} \quad (4)$$

where $\#\{\Delta g^{-1} \cap W\}$ denotes the number of elements in the set $\Delta g^{-1} \cap W$. Since $x \in \Delta g^{-1} \cap W \Leftrightarrow xg \in \Delta$, and $x \in W$, we get

$$xg(D) \subset C \text{ i.e., } xg(v) \in C \quad (\because D = \mathbb{R}_{>0}v).$$

Now we claim that $\#\{\Delta g^{-1} \cap W\} \leq 1$. Otherwise, $\exists x_1, x_2 \in \Delta g^{-1} \cap W$ such that $x_1 \neq x_2$. We have

$$\begin{aligned} x_1g(v) &= c_1 \in C \text{ and } x_2g(v) = c_2 \in C \\ \Rightarrow x_2x_1^{-1}(c_1) &= x_2x_1^{-1}(x_1(gv)) = x_2(gv) = c_2 \\ \Rightarrow x_2x_1^{-1}(C) \cap C &\neq \emptyset \\ \Rightarrow x_2x_1^{-1} &= 1 \quad (\because x_2x_1^{-1} \in W) \\ \Rightarrow x_2 &= x_1, \end{aligned}$$

which is a contradiction to our assumption. Therefore $\#\{\Delta g^{-1} \cap W\} \leq 1$, and we get

$$\begin{aligned} \int_{W \setminus G} \#\{\Delta g^{-1} \cap W\} d\nu(Wg) &\leq \int_{W \setminus G} d\nu(Wg) \\ &= \nu(W \setminus G). \end{aligned} \quad (5)$$

By (3), (4) and (5), we get

$$\int_G f \, d\mu \leq \nu(W \setminus G).$$

As $f \prec \Delta$ was chosen arbitrarily, we get $\mu(\Delta) \leq \nu(W \setminus G)$ (by using (2)), and hence $\mu(\Delta) < \infty$. \square

Now we prove Theorem 1.6 using the above lemmas. We have B , a non-degenerate bilinear form on V . Let G be the group of real points of the orthogonal group of B and μ be a Haar measure on G . It is clear that the group G is unimodular and contains W . Also, we can identify V with its dual V^* by means of B ; in particular, we denote by (e_i^*) the basis of V dual to the basis (e_i) , and by C the interior of the simplicial cone \bar{C} generated by the (e_i^*) . Since W is infinite, the bilinear form B is not positive definite and it has the signature (p, q) , where $p + q = n$ and $p, q \geq 1$. We prove few more lemmas to prove Theorem 1.6.

Lemma 2.7. $B(v, v) \neq 0$, for some $v \in C$.

Proof. Since for any $v \in C$, $C - v$ is an open subset of V containing the origin 0 (since C is an open subset of V), V is generated by $C - v$ (as an abelian group). In particular, $C - v$ generates V as a vector space over \mathbb{R} , therefore there exists $\{v_1 - v, v_2 - v, \dots, v_n - v\}$ a basis of V over \mathbb{R} contained in $C - v$, where $v_i \in C, \forall 1 \leq i \leq n$. Now if possible, let $B(v, v) = 0, \forall v \in C$.

$$\begin{aligned} \Rightarrow B(u, v) &= \frac{1}{2}(B(u + v, u + v) - B(u, u) - B(v, v)) \\ &= 0 \quad \forall u, v \in C \quad (\because \forall u, v \in C, u + v \in C). \end{aligned} \quad (6)$$

Now we show that if $B(v, v) = 0, \forall v \in C$, then $B \equiv 0$, which gives a contradiction (since B is non-zero). Since $v_i, v \in C$, using the bilinearity of B and (6), we get

$$B(v_i - v, v_j - v) = 0, \quad \forall 1 \leq i, j \leq n$$

i.e., $B \equiv 0$. Therefore $\exists v \in C$ such that $B(v, v) \neq 0$. \square

Let $v \in C$ be an element for which $B(v, v) \neq 0$. Let $L_v = \{u \in V \mid B(u, v) = 0\}$. Since $B(v, v) \neq 0$, $V = \mathbb{R}v \oplus L_v$. Now take $D = \mathbb{R}_{>0}v \subset C$ a half line contained in C . The group $G = O(B)(\mathbb{R}) \leq GL(n, \mathbb{R})$ is unimodular with a Haar measure μ and it contains the Coxeter group W as a discrete subgroup. Let ν be a G -invariant measure on the quotient $W \backslash G$ such that $\nu(W \backslash G) < \infty$ i.e., W is a lattice in G . We have a basis $\{v, u_1, u_2, \dots, u_{n-1}\}$ of V over \mathbb{R} , where $\{u_1, u_2, \dots, u_{n-1}\}$ is a basis of L_v over \mathbb{R} . With respect to this basis of V , $B = B_1 \oplus B_2$, where $B_1 = B|_{\mathbb{R}v}$ and $B_2 = B|_{L_v}$. The symmetric matrix associated to the bilinear form B , with respect to this basis, is

of the form

$$B = \begin{pmatrix} B_1(v, v) & 0 & 0 & \dots & 0 \\ 0 & & & & \\ 0 & B_2 & & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}.$$

Let $H = O(B_2)(\mathbb{R}) \leq GL(L_v) (= GL(n-1, \mathbb{R}))$ be the orthogonal group of the bilinear form B_2 on L_v . It is clear that

$$G' = \left\{ \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & h & \\ 0 & & & \end{pmatrix} : h \in H \right\}$$

is a closed subgroup of G and $\forall g \in G'$, $g(v) = v$ i.e., G' is a closed subgroup of G_D , therefore it is compact (by Lemma 2.6). Also, G' is isomorphic as a Lie group to $H = O(B_2)(\mathbb{R})$, therefore H is a compact subgroup of $GL(L_v)$. It shows that the bilinear form B_2 is either positive definite or negative definite. Since the group W is infinite, the bilinear form B can not be positive or negative definite. Therefore B has the signature $(n-1, 1)$ or $(1, n-1)$.

Now we show that B can not have the signature $(1, n-1)$.

Lemma 2.8. *If there is a relation $(s_i s_j)^{m_{i,j}} = 1$, for some $i \neq j$ and $2 \leq m_{i,j} < \infty$ in the generators of the Coxeter group W and the bilinear form B as above, then B has the signature $(n-1, 1)$.*

Proof. For $2 \leq m_{i,j} < \infty$, $B(e_i, e_j) = -\cos\left(\frac{\pi}{m_{i,j}}\right) > -1$, and hence

$$\begin{aligned} B(\lambda e_i + \delta e_j, \lambda e_i + \delta e_j) &= \lambda^2 B(e_i, e_i) + \delta^2 B(e_j, e_j) + 2\lambda\delta B(e_i, e_j) \\ &= \lambda^2 + \delta^2 + 2\lambda\delta B(e_i, e_j) \\ &> \lambda^2 + \delta^2 - 2\lambda\delta \quad (\because B(e_i, e_j) > -1) \\ &= (\lambda - \delta)^2 \\ &\geq 0. \end{aligned}$$

Therefore $\forall (\lambda, \delta) \neq (0, 0); \lambda, \delta \in \mathbb{R}, B(\lambda e_i + \delta e_j, \lambda e_i + \delta e_j) > 0$. Let $V_{i,j} = \mathbb{R}e_i \oplus \mathbb{R}e_j$ be a subspace of V . The restriction of the bilinear form B on $V_{i,j}$ is non-degenerate and positive definite. Therefore $V = V_{i,j} \oplus V_{i,j}^\perp$, and with respect to a basis of V which is the union of a

basis of $V_{i,j}$ and a basis of $V_{i,j}^\perp$, the matrix of the bilinear form B is

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & B|_{V_{i,j}^\perp} & \\ 0 & 0 & & & \end{pmatrix}$$

and $B|_{V_{i,j}^\perp}$ is non-degenerate. The above matrix form of the bilinear form B shows that its signature is (p, q) , where $p, q \in \mathbb{N}$, $p + q = n$, and $p \geq 2$. Therefore the possibility for the signature of B to be $(1, n-1)$ is excluded i.e., B has the signature $(n-1, 1)$. \square

Lemma 2.9. *If $(s_i s_j)^\infty = 1$, for $i \neq j$ and $s_i s_i = 1, \forall i, j \in \{1, 2, \dots, n\}$ are the only relations in the generators of the Coxeter group W and the bilinear form B as above, then B has the signature $(n-1, 1)$.*

Proof. These relations mean that all the vertices in the Coxeter graph of the Coxeter group W are joined by an edge of weight ∞ , and $B(e_i, e_i) = 1$, and $B(e_i, e_j) = -1$, for $i \neq j$. These relations are not possible in a Coxeter group W with 2 generators ($\because B$ is non-degenerate). Since all the vertices are joined by an edge in the Coxeter graph, the Coxeter graph will contain a triangle for each $n \geq 3$. Let s_1, s_2 and s_3 are any three vertices which are joined to each other to form a triangle. Let $V_1 = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$ be a subspace of V , and $B_1 = B|_{V_1}$ be a bilinear form on V_1 . Now we show that B_1 has the signature $(2, 1)$ and it shows that $V = V_1 \oplus V_1^\perp$ i.e., the signature of B is (p, q) with $p \geq 2$. The matrix form of B_1 with respect to the basis $\{e_1, e_2, e_3\}$ of V_1 over \mathbb{R} is

$$B_1 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Also, $\det(B_1) = 1(1-1) + 1(-1-1) - (1+1) = -4 \neq 0$, therefore B_1 is non-degenerate.

Now we compute the eigenvalues of B_1 . If $\lambda \in \mathbb{R}$ is an eigenvalue of the matrix

$$B_1 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

then $\det(B_1 - \lambda I_3) = 0$, where I_3 is the 3×3 identity matrix.

$$\begin{aligned}
& \det(B_1 - \lambda I_3) = 0 \\
& \Rightarrow (1 - \lambda)((1 - \lambda)^2 - 1) + 1(-(1 - \lambda) - 1) - 1(1 + (1 - \lambda)) = 0 \\
& \Rightarrow (1 - \lambda)\lambda(\lambda - 2) + 2(\lambda - 2) = 0 \\
& \Rightarrow (\lambda - 2)(-\lambda^2 + \lambda + 2) = 0 \\
& \Rightarrow -1(\lambda - 2)(\lambda - 2)(\lambda + 1) = 0 \\
& \Rightarrow \lambda = 2, 2, -1 \text{ are the eigenvalues of the matrix } B_1.
\end{aligned}$$

Since a symmetric matrix is orthogonally diagonalizable, we get the signature of the bilinear form B_1 is $(2, 1)$. It shows that the possibility for the signature of the bilinear form B to be $(1, n - 1)$ is excluded. Therefore the signature of the bilinear form B is $(n - 1, 1)$. \square

Since we had $V = \mathbb{R}v \oplus L_v$, where $v \in C$ is an element for which $B(v, v) \neq 0$, and $L_v = \{u \in V \mid B(u, v) = 0\}$, the condition on the signature of B forces $B(v, v) < 0$ ($\because B|_{L_v}$ is positive definite and B is non-degenerate and non-positive). The above proof also shows that if $B(u, u) \neq 0$, for any $u \in C$, then $B(u, u) < 0$. Now we show that $B(u, u) \neq 0$, for any $u \in C$. Otherwise $\exists u \in C$ such that $B(u, u) = 0$. Since the bilinear form B is non-degenerate, $\exists v \in V$ such that $B(v, v) = 0$, and $B(u, v) = 1$ (see Theorem 6.10 of [5]). Also, for any $\alpha, \beta > 0$ in \mathbb{R} , $B(\alpha u + \beta v, \alpha u + \beta v) = 2\alpha\beta > 0$. Since $u \in C$, and C is open in V , $\exists \alpha, \beta > 0$ in \mathbb{R} such that $\alpha u + \beta v \in C$, and $B(\alpha u + \beta v, \alpha u + \beta v) = 2\alpha\beta > 0$, which is a contradiction. Therefore we get $B(u, u) \neq 0, \forall u \in C$. Hence $B(u, u) < 0, \forall u \in C$. \square

3. EXAMPLE

In this section we will do some computations and show that a right angled Coxeter group W generated by 3 elements is isomorphic to a lattice in the group $O(B)(\mathbb{R}) = O(2, 1)$ of real rank 1. Let W be the right angled Coxeter group generated by 3 elements s_1, s_2 , and s_3 with the relations: $(s_i s_j)^{m_{i,j}} = 1$, where $m_{i,i} = 1, \forall i \in \{1, 2, 3\}$, and $m_{1,2} = m_{2,3} = \infty, m_{1,3} = 2$. Let \mathbb{R}^3 be a 3-dimensional vector space over \mathbb{R} with a basis $\{e_1, e_2, e_3\}$. We define a symmetric bilinear form B on \mathbb{R}^3 as

$$B(e_i, e_j) = -\cos\left(\frac{\pi}{m_{i,j}}\right), \quad \text{for } m_{i,j} \neq \infty,$$

and for $m_{i,j} = \infty$, we define $B(e_i, e_j) = -1$. With respect to the basis $\{e_1, e_2, e_3\}$, the matrix of B is

$$B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

One can check that the bilinear form B is non-degenerate. Now we define a representation $\rho : W \rightarrow \text{GL}(\mathbb{R}^3)$ as $\rho(s_i)(e_j) = e_j - 2B(e_j, e_i)e_i$. It can be checked easily that ρ maps the group W inside the orthogonal group $\text{O}(B)(\mathbb{R})$ of the bilinear form B . We will show in this example that the group W by this representation ρ is mapped onto a finite index subgroup of $\text{O}(B)(\mathbb{Z})$, the group of integral points of the orthogonal group $\text{O}(B)$ of the bilinear form B , and it shows that the group W is a lattice in $\text{O}(B)(\mathbb{R})$.

With respect to the basis $\{e_1, e_2, e_3\}$, the matrices of $\rho(s_1)$, $\rho(s_2)$ and $\rho(s_3)$ are

$$\rho(s_1) = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \rho(s_2) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \rho(s_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}.$$

If we do some integral change in the basis of \mathbb{R}^3 over \mathbb{R} , and take $\{e_1 + e_2, e_2, e_2 + e_3\}$ as a basis of \mathbb{R}^3 , then the corresponding matrices of $\rho(s_1), \rho(s_2), \rho(s_3)$ and B , become

$$\rho(s_1) = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \rho(s_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \rho(s_3) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Also,

$$\rho(s_2 s_1) = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \rho(s_2 s_3) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}.$$

It shows that the signature of the bilinear form B is $(2, 1)$ (since, its roots are 1, 1, -1).

The adjoint representation of $\text{SL}(2, \mathbb{R})$ on its Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, maps the group $\text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{\pm I\}$ isomorphically onto its image and it preserves the killing form K defined on $\mathfrak{sl}(2, \mathbb{R})$. The Lie

algebra $\mathfrak{sl}(2, \mathbb{R})$ can be identified with \mathbb{R}^3 as a vector space over \mathbb{R} , with the basis

$$\left\{ e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

The killing form K on $\mathfrak{sl}(2, \mathbb{R})$ is defined by

$$K(X, Y) = \frac{1}{2} \text{tr}(XY), \quad \forall \quad X, Y \in \mathfrak{sl}(2, \mathbb{R}).$$

If we do some integral change in the basis of $\mathfrak{sl}(2, \mathbb{R})$ over \mathbb{R} and take

$$\left\{ \epsilon_1 = -2e_1 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \epsilon_2 = e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \epsilon_3 = e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

as a basis of $\mathfrak{sl}(2, \mathbb{R})$ over \mathbb{R} , then the matrix of K becomes

$$K = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Therefore the bilinear form B associated to the Coxeter group W , is equivalent to the killing form K on $\mathfrak{sl}(2, \mathbb{R})$ over \mathbb{Z} , and the signature of K is also $(2, 1)$. Hence the group $\text{SL}(2, \mathbb{R})/\{\pm I\}$ maps into $\text{O}(2, 1) \leq \text{GL}(3, \mathbb{R})$, by the adjoint representation Ad of $\text{SL}(2, \mathbb{R})$ on its Lie algebra, where $\text{O}(2, 1) = \text{O}(B)(\mathbb{R})$. Since the group $\text{SL}(2, \mathbb{R})/\{\pm I\}$ is connected, it is mapped inside the connected component $\text{SO}(2, 1)$ of $\text{O}(2, 1)$ of the identity element. In fact, $\text{Ad}(\text{SL}(2, \mathbb{R})/\{\pm I\}) = \text{SO}(2, 1)$ ($\because \dim \text{SL}(2, \mathbb{R})/\{\pm I\} = \dim \text{SO}(2, 1)$) i.e., $\text{SL}(2, \mathbb{R})/\{\pm I\} \cong \text{SO}(2, 1)$. Hence $\text{SL}(2, \mathbb{Z})/\{\pm I\}$ is a lattice in $\text{SO}(2, 1)$. In fact, $\text{SL}(2, \mathbb{Z})/\{\pm I\}$ is a lattice in $\text{O}(2, 1)$ ($\because \text{SO}(2, 1)$ has finite index in $\text{O}(2, 1)$).

The right-angled Coxeter group W is mapped inside $\text{O}(B)(\mathbb{Z}) = \text{O}(2, 1)(\mathbb{Z})$, by the representation ρ . We construct a finite index subgroup H of $\text{SL}(2, \mathbb{Z})/\{\pm I\}$ which preserves a lattice L in $\mathfrak{sl}(2, \mathbb{R}) = \mathbb{R}^3$ (as a vector space) i.e., H is also mapped inside $\text{O}(2, 1)(\mathbb{Z})$, by the representation Ad , and being a finite index subgroup of $\text{SL}(2, \mathbb{Z})/\{\pm I\}$, H becomes a lattice in $\text{O}(2, 1)$. Also, we construct a finite index subgroup H' of W which is mapped onto $\text{Ad}(H)$, by the representation ρ , and hence $\rho(H')$ becomes a lattice in $\text{O}(2, 1)$, and W becomes a finite index subgroup of $\text{O}(2, 1)(\mathbb{Z})$ i.e., a lattice in $\text{O}(2, 1)$.

Lemma 3.1. *The group $\text{SL}(2, \mathbb{Z})/\{\pm I\}$ is generated by $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$*

and $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and it has a presentation as $\langle w, x; w^2, (wx)^3 \rangle$ i.e., it is the free product of the cyclic group of order 2 generated by w and the cyclic group of order 3 generated by wx .

For a proof, see Theorem 2 and the preceding remark of Chapter VII of [8].

We get $x^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $w x^2 w^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1}$.

Let H be the subgroup of $SL(2, \mathbb{Z})/\{\pm I\}$ generated by $\{x^2, w x^2 w^{-1}\}$. It can be shown using the presentation of $SL(2, \mathbb{Z})/\{\pm I\}$ as in the above lemma, that the subgroup H has finite index in $SL(2, \mathbb{Z})/\{\pm I\}$. Also, one can show easily that H preserves the lattice

$$L = \mathbb{Z} \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{Z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{Z} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

in $sl(2, \mathbb{R})$. Hence H is mapped inside $O(2, 1)(\mathbb{Z})$, by the adjoint representation Ad , and being a lattice (·: it has finite index in $SL(2, \mathbb{Z})/\{\pm I\}$) in $O(2, 1)(\mathbb{R})$, it has finite index in $O(2, 1)(\mathbb{Z})$.

By an easy computation, we find that the matrices of $Ad(x^2)$ and $Ad(w x^2 w^{-1})^{-1}$ in $O(2, 1)(\mathbb{R})$ w.r.t. the basis

$$\left\{ \epsilon_1 = -2e_1 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \epsilon_2 = e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \epsilon_3 = e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\},$$

are

$$Ad(x^2) = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } Ad(w x^2 w^{-1})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & 4 & 1 \end{pmatrix}. \quad (7)$$

Let H' be the subgroup of the Coxeter group W generated by the set $\{s_2 s_1, s_2 s_3\}$. It can be shown easily that the subgroup H' has finite index in the group W . We find that the matrices of $\rho(s_2 s_1)$ and $\rho(s_2 s_3)$ in $O(2, 1)(\mathbb{R})$ w.r.t. the basis $\{e_1 + e_2, e_2, e_2 + e_3\}$, are

$$\rho(s_2 s_1) = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \rho(s_2 s_3) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}. \quad (8)$$

Also, $\rho(s_2 s_3)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & 4 & 1 \end{pmatrix} = Ad(w x^2 w^{-1})^{-1}$, and hence by (7) and

(8), we see that H is a subgroup of H' . Therefore H' is a finite index subgroup of $O(2, 1)(\mathbb{Z})$, and hence the Coxeter group W is also a finite index subgroup of $O(2, 1)(\mathbb{Z})$ i.e., W is a lattice in $O(2, 1)$.

4. PROOF OF THEOREM 1.4

Let $O(B)$ be the orthogonal group of the bilinear form B and $O(p, q)$ be the group of real points of the group $O(B)$ i.e., $O(p, q) = O(B)(\mathbb{R})$,

where (p, q) is the signature of B with $p, q \geq 1$, and $p + q = n$. Let $\mathrm{SO}(B)$ be the connected component of the identity element of $\mathrm{O}(B)$, and $\mathrm{SO}(p, q) = \mathrm{SO}(B)(\mathbb{R})$. The subgroup $\mathrm{SO}(p, q)$ has finite index (four) in the group $\mathrm{O}(p, q)$, therefore any finite index subgroup L' of the Coxeter group W contains a finite index subgroup $L \leq \mathrm{SO}(p, q)$, namely $L = L' \cap \mathrm{SO}(p, q)$. If L' is isomorphic to an irreducible lattice Γ' in a semisimple group H of \mathbb{R} -rank ≥ 2 , then L will be isomorphic to a finite index subgroup Γ of Γ' . Also, it can be shown easily that a finite index subgroup Γ of an irreducible lattice Γ' is an irreducible lattice in H . We prove few lemmas which will be used in the proof of Theorem 1.4.

Lemma 4.1. *There exists a connected semisimple adjoint group \tilde{G} and an (central) isogeny $\pi : \mathrm{SO}(B) \longrightarrow \tilde{G}$.*

For a proof, see Theorem 2.6 of [7].

In fact, \tilde{G} is an \mathbb{R} -simple group (since the group $\mathrm{SO}(B)$ has maximal normal subgroup $\{\pm I\}$ which is the center of $\mathrm{SO}(B)$ and π is central therefore the kernel of π is $\{\pm I\}$).

Lemma 4.2. *If L is a discrete subgroup of $\mathrm{SO}(B)(\mathbb{R}) = \mathrm{SO}(p, q)$, then $\pi(L)$ is a discrete subgroup of $\tilde{G}(\mathbb{R})$.*

Proof. The homomorphism π is an open map and its kernel is finite. Now using the discreteness of L , it can be shown easily that $\pi(L)$ is a discrete subgroup of $\tilde{G}(\mathbb{R})$. \square

Lemma 4.3. *If L is a Zariski dense subgroup of $\mathrm{SO}(B)$, then $\pi(L)$ is a Zariski dense subgroup of \tilde{G} .*

Proof. Since the map $\pi : \mathrm{SO}(B) \longrightarrow \tilde{G}$ is continuous w.r.t. the Zariski topology, we get $\pi(\overline{L}) \subseteq \overline{\pi(L)}$. Therefore $\overline{\pi(L)} = \tilde{G}$ (since $\overline{L} = \mathrm{SO}(B)$). \square

Lemma 4.4. $\mathbb{R}\text{-rank}(\mathrm{SO}(B)) = \mathbb{R}\text{-rank}(\tilde{G})$.

Proof. The group $\mathrm{SO}(B)$ has an \mathbb{R} -split torus T i.e., all the characters $\chi : T \longrightarrow G_m$ are defined over \mathbb{R} . We claim that the subgroup $\pi(T)$ of \tilde{G} is an \mathbb{R} -split torus. For, it is clear that $\pi(T)$ is a connected, abelian subgroup of \tilde{G} . Also, $\pi(T)$ is diagonalizable over \mathbb{C} (since under a homomorphism of algebraic groups, torus maps to a torus). To show $\pi(T)$ is \mathbb{R} -split, we show that all the characters $\chi : \pi(T) \longrightarrow G_m$ are defined over \mathbb{R} . For, let us define $\chi' : T \longrightarrow G_m$ as $\chi'(t) = \chi(\pi(t))$. It is clear that χ' is a character of the torus T which is \mathbb{R} -split, therefore χ' is defined over \mathbb{R} . Now we show that χ is fixed under the action

of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on $\text{Hom}(T, G_m)$ i.e., χ is defined over \mathbb{R} . For, let $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$. We have

$$\begin{aligned}
 \chi(\pi(t)) &= \chi'(t) \\
 &= (\sigma.\chi')(t) \\
 &= \sigma(\chi'(\sigma^{-1}t)) \\
 &= \sigma(\chi \circ \pi(\sigma^{-1}t)) \\
 &= \sigma(\chi\sigma^{-1}(\sigma.\pi)(t)) \\
 &= (\sigma.\chi)(\pi(t)) \quad (\because \chi' \text{ and } \pi \text{ are defined over } \mathbb{R}).
 \end{aligned}$$

Since the above equality is true for all $t \in T$, and π is surjective, therefore we get $\sigma.\chi = \chi$, for all $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$. Hence all the characters $\chi : \pi(T) \rightarrow G_m$ are defined over \mathbb{R} i.e., $\pi(T)$ is an \mathbb{R} -split torus in \tilde{G} . Also, π has finite kernel, therefore $\mathbb{R}\text{-rank}(\tilde{G}) = \mathbb{R}\text{-rank}(\text{SO}(B))$. \square

Theorem 4.5. *Let L be a discrete subgroup of the group $\text{SO}(p, q)$. Let H be a connected semisimple Lie group without non-trivial compact factor groups, of real rank ≥ 2 with trivial center. Let $\Gamma \leq H$ be an irreducible lattice and $\delta : \Gamma \rightarrow L \leq \text{SO}(B)(\mathbb{R}) = \text{SO}(p, q)$ be an isomorphism and $\delta(\Gamma) = L$ is Zariski dense in $\text{SO}(B)$. Let \tilde{G} be a connected semisimple adjoint group with an (central) isogeny $\pi : \text{SO}(B) \rightarrow \tilde{G}$. Let $\delta' : \Gamma \rightarrow \pi(L) \leq \tilde{G}(\mathbb{R})$ be a continuous homomorphism defined as $\delta' = \pi \circ \delta$. Let \tilde{G} has no nontrivial \mathbb{R} -anisotropic factors. Then δ' extends uniquely to an isomorphism $\tilde{\delta}' : H \rightarrow \tilde{G}(\mathbb{R})^\circ$, and the group $\tilde{G}(\mathbb{R})$ has $\mathbb{R}\text{-rank} \geq 2$, and $\pi(L)$ is a lattice in $\tilde{G}(\mathbb{R})$.*

Proof. The group \tilde{G} is adjoint, and has no nontrivial \mathbb{R} -anisotropic factors and $\pi(L)$ is a discrete subgroup of $\tilde{G}(\mathbb{R})$ (by Lemma 4.2), and it is also Zariski dense in \tilde{G} (by Lemma 4.3). Therefore by Theorem 1.7 we get a continuous homomorphism $\tilde{\delta}' : H \rightarrow \tilde{G}(\mathbb{R})$ with $\tilde{\delta}'|_\Gamma = \delta'$. Since the group $\tilde{\delta}'(H)$ is a connected semisimple group which is Zariski dense in \tilde{G} (since $\tilde{\delta}'(\Gamma) = \pi(L)$ is Zariski dense in \tilde{G}), it follows from [6] (Remark 6.17 (ii) of Chapter IX), that $\tilde{\delta}'(H) = \tilde{G}(\mathbb{R})^\circ$. Since H has trivial center and no nontrivial compact factor groups, Γ is an irreducible lattice in H , and $\delta'(\Gamma) = \pi(L)$ is a nontrivial discrete subgroup of $\tilde{G}(\mathbb{R})$, therefore it follows from [6] (Remark 6.17 (iii) of Chapter IX) that $\tilde{\delta}'$ is an isomorphism of H onto $\tilde{G}(\mathbb{R})^\circ$, and hence $\pi(L)$ is a lattice in $\tilde{G}(\mathbb{R})^\circ$, and the \mathbb{R} -rank of $\tilde{G}(\mathbb{R})$ is ≥ 2 . Since $\tilde{G}(\mathbb{R})^\circ$ is a finite index subgroup of $\tilde{G}(\mathbb{R})$, $\pi(L)$ is a lattice in $\tilde{G}(\mathbb{R})$. \square

Remark. In the proof of Theorem 4.5, the fact that H has trivial center, has been used only to show that $\tilde{\delta}'$ is an isomorphism. If the

group H does not have trivial center, then the homomorphism $\tilde{\delta}'$ has finite kernel, and $\tilde{\delta}'(\Gamma) = \pi(L)$ is still a lattice in $\tilde{G}(\mathbb{R})$ (since under such homomorphism $\tilde{\delta}'$, a lattice maps onto a lattice). Therefore Theorem 4.5 is also true for a connected semisimple Lie group with non-trivial center, and without non-trivial compact factor groups, of real rank ≥ 2 .

Lemma 4.6. *Let L be a discrete subgroup of $SO(p, q)$ and \tilde{G}, π as in Lemma 4.1. If $\pi(L)$ is a lattice in $\tilde{G}(\mathbb{R})$, then L is a lattice in $SO(p, q)$.*

Proof. Since L is a discrete subgroup of $SO(p, q)$ and $SO(p, q)$ is unimodular, the quotient $L \backslash SO(p, q)$ has an $SO(p, q)$ -invariant measure μ . The homomorphism $\pi : SO(p, q) \rightarrow \tilde{G}(\mathbb{R})$ induces a continuous map $\tilde{\pi} : L \backslash SO(p, q) \rightarrow \pi(L) \backslash \tilde{G}(\mathbb{R})$, which is defined as $\tilde{\pi}(Lg) = \pi(L)\pi(g)$. It can be checked easily that the pushforward measure $\tilde{\pi}_*(\mu)$ on the quotient $\pi(L) \backslash \tilde{G}(\mathbb{R})$ defined as $\tilde{\pi}_*(\mu)(\tilde{E}) = \mu(\tilde{\pi}^{-1}(\tilde{E}))$, for all measurable subsets \tilde{E} of $\pi(L) \backslash \tilde{G}(\mathbb{R})$, is $\tilde{G}(\mathbb{R})$ -invariant (since $\tilde{\pi}$ is surjective and μ is $SO(p, q)$ -invariant). Therefore by the uniqueness of a $\tilde{G}(\mathbb{R})$ -invariant measure on the quotient $\pi(L) \backslash \tilde{G}(\mathbb{R})$, we get $\tilde{\pi}_*(\mu)(\pi(L) \backslash \tilde{G}(\mathbb{R})) < \infty$ (since $\pi(L)$ is a lattice in $\tilde{G}(\mathbb{R})$), and hence $\mu(L \backslash SO(p, q)) < \infty$ i.e., L is a lattice in $SO(p, q)$. \square

Theorem 4.7. *The Coxeter group W is Zariski dense in the group $O(B)$.*

For a proof, see [1].

Lemma 4.8. *Let G be a topological group and L', L are subgroups of G such that L has finite index in L' . Then $(\bar{L}')^o = (\bar{L})^o$, where $(\bar{L})^o$ is the connected component of the identity element of the closure of L in G .*

Proof. Since L has finite index d (say) in L' ,

$$\begin{aligned} L' &= \cup_{i=1}^d \gamma_i L; \quad \gamma_i \in L' \\ \Rightarrow \bar{L}' &= \cup_{i=1}^d \gamma_i \bar{L}; \quad \gamma_i \in L' \\ \Rightarrow [\bar{L}' : \bar{L}] &\leq d \\ \Rightarrow \bar{L} &\text{ is a finite index subgroup of the group } \bar{L}'. \end{aligned}$$

Hence \bar{L} is closed and open in \bar{L}' and $(\bar{L}')^o \supset (\bar{L})^o$, therefore $(\bar{L})^o$ is open and closed in $(\bar{L}')^o$ which is connected. This shows that $(\bar{L}')^o = (\bar{L})^o$. \square

Corollary 4.9. *In the above lemma if we take $G = O(p, q) = O(B)(\mathbb{R})$, and $L' = W$, the Coxeter group and $L \leq SO(p, q) \cap W$ such that $[W : L] < \infty$, then $\bar{L} = SO(p, q)$ i.e., L is Zariski dense in $SO(p, q)$. Hence*

L is Zariski dense in $\mathrm{SO}(B)$ ($\because \mathrm{SO}(B)(\mathbb{R}) = \mathrm{SO}(p, q)$ is Zariski dense in $\mathrm{SO}(B)$).

Proof. Just use Theorem 4.7 and Lemma 4.8. \square

Lemma 4.10. *If L is a lattice in $\mathrm{SO}(p, q)$, then L is also a lattice in $\mathrm{O}(p, q)$.*

Proof. Since $\mathrm{O}(p, q)$ is unimodular and L is a discrete subgroup of $\mathrm{O}(p, q)$, $L \backslash \mathrm{O}(p, q)$ has a nonzero $\mathrm{O}(p, q)$ -invariant measure μ . Since $\mathrm{SO}(p, q)$ is open in $\mathrm{O}(p, q)$, its Borel σ -algebra is a subalgebra of the Borel σ -algebra of $\mathrm{O}(p, q)$ and the restriction of μ on $L \backslash \mathrm{SO}(p, q)$ is a nonzero $\mathrm{SO}(p, q)$ -invariant measure. Now to show that L is a lattice in $\mathrm{O}(p, q)$, we claim that $\mu(L \backslash \mathrm{O}(p, q)) < \infty$. For,

$$L \backslash \mathrm{O}(p, q) = \{Lg | g \in \mathrm{O}(p, q)\},$$

and

$$\mathrm{O}(p, q) = \{\mathrm{SO}(p, q)g_i | g_i \in \mathrm{O}(p, q), 1 \leq i \leq 4\}.$$

For each $g \in \mathrm{O}(p, q)$, $\exists h \in \mathrm{SO}(p, q)$ such that $g = hg_i$, for some $1 \leq i \leq 4$. Therefore $Lg = Lhg_i \in (L \backslash \mathrm{SO}(p, q))g_i$, and

$$L \backslash \mathrm{O}(p, q) = \cup_{i=1}^4 (L \backslash \mathrm{SO}(p, q))g_i.$$

$$\begin{aligned} \Rightarrow \mu(L \backslash \mathrm{O}(p, q)) &\leq \sum_{i=1}^4 \mu(L \backslash \mathrm{SO}(p, q)g_i) \\ &= \sum_{i=1}^4 \mu(L \backslash \mathrm{SO}(p, q)) \\ &< \infty. \end{aligned}$$

It shows that L is a lattice in $\mathrm{O}(p, q)$. \square

From the remark at the beginning of this section and Corollary 4.9, it follows that if the Coxeter group W contains a finite index subgroup $L \leq \mathrm{SO}(p, q)$, which is isomorphic to an irreducible lattice in a connected semisimple Lie group H without nontrivial compact factor groups, of real rank ≥ 2 , then $\mathrm{SO}(p, q)$ has real rank ≥ 2 (by Lemma 4.4 and Theorem 4.5) i.e., $p, q \geq 2$, and L is a lattice in $\mathrm{SO}(p, q)$ (by Theorem 4.5 and Lemma 4.6). Lemma 4.10 shows that L is a lattice in $\mathrm{O}(p, q)$ also, and hence W becomes a lattice in $\mathrm{O}(p, q)$ (since a discrete subgroup W of a Lie group G which contains a lattice L , is a lattice in G). This is a contradiction to Theorem 1.6 which has been proved in Section 2. Hence Theorem 1.4 is proved. \square

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